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1 set identities

as you might have noticed, a lot of operations that we do with sets are similar to the logical operations that we have before; like conjunction ($\wedge$) and intersection ($\cap$), disjunction ($\vee$) and union ($\cup$), negation ($\neg$) and complement ($\bar{S}$); and just as how logical equivalences help us to manipulate logical expressions, **set identities** help us simplify and understand complex set definitions.

1.1 important identities

here are some notable identities of sets; you might find similarity from many of them, when compared to the logical equivalences we learned earlier. keep in mind that the set U is a universal set.

name	identity
identity laws	$A \cup \emptyset = A$ $A \cap U = A$
domination laws	$A \cup U = U$ $A \cap \emptyset = \emptyset$
idempotent laws	$A \cup A = A$ $A \cap A = A$
complementation law	$\bar{\bar{A}} = A$
commutative laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
demorgan's laws	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$
absorption laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
complement laws	$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$

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1.2 proving set identities

there are four common methods to prove set identities.

1. membership tables
    - similar to how we use a truth table to prove logical equivalence, membership tables prove set equivalences.
2. logical argument ("mutual subset" method)
    - we prove one set (A) is a subset of (B), and B is a subset of A. it is similar to how we prove biconditionals using logical equivalences.
3. using set builder notation
    - we apply set identities and logical equivalences using set builder notation to prove set identities.

- applying other known set identities
    - it is similar to using existing logical equivalences to prove new ones.

1.2.1 membership tables

the membership table indicate the ways in which an arbitrary element may or may not be included between sets and the set operation.

A	B	$A \cap B$ (logical $\land$)
1	1	1
1	0	0
0	1	0
0	0	0

1.2.1.1 example

goal: prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
proof: using membership table.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

since the appropriate columns of the membership table are the same, we can conclude that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ■

1.2.2 logical argument (mutual subset)

you might recall that: $A = B \Leftrightarrow (A \subseteq B) \land (B \subseteq A)$. as a result, we can prove a set's identity by proving that each side of the equation is a subset of each other.

1.2.2.1 example

goal: prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
proof: we have to prove two things: $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$.

part one: $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$
1. let x be an arbitrary element from $\overline{A \cap B}$.
2. by definition of complement, $x \notin (A \cap B)$.
3. by definition of $\notin$, $\neg(x \in A \cap B)$
4. by definition of intersection, $\neg(x \in A \land x \in B)$
5. by demorgan's law, $\neg(x \in A) \lor \neg(x \in B)$
    - in the first case, $x \notin A$, so by definition of a complement, $x \in \bar{A}$
    - in the second case, $x \notin B$, and by definition of a complement, $x \in \bar{B}$
    - combining both cases, $x \in \bar{A} \lor x \in \bar{B}$
6. by definition of a union, $x \in \bar{A} \cup \bar{B}$

part two: $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$
1. let x be an arbitrary element from $x \in \bar{A} \cup \bar{B}$
2. by definition of union, $x \in \bar{A} \lor x \in \bar{B}$
3. by definition of complement, $x \notin A \lor x \notin B$
    - in the first case, $x \notin A$, so x cannot be in both A and B
    - in the second case, $x \notin B$, so x cannot be in both A and B
4. therefore, if $x \in \bar{A} \cup \bar{B}$, then $x \in \overline{A \cap B}$

since we haven shown both $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$, we can conclude that $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

1.2.3 set builder notation

we can use set builder notation to make very precise proofs. it is similar to the logical argument method, however, it uses less words and more mathematical notations.

1.2.3.1 example

goal: prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$
proof:

$$\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin (A \cap B)\} && \text{definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{definition of } \notin \\
&= \{x \mid \neg(x \in A \land x \in B)\} && \text{definition of intersect} \\
&= \{x \mid \neg(x \in A) \lor \neg(x \in B)\} && \text{demorgan's law} \\
&= \{x \mid x \notin A \lor x \notin B\} && \text{definition of } \notin \\
&= \left\{x \mid x \in \bar{A} \lor x \in \bar{B}\right\} && \text{definition of complement} \\
&= \left\{x \mid x \in \bar{A} \cup \bar{B}\right)\} && \text{definition of union}
\end{aligned}$$

1.2.4 apply set identities

we can also directly manipulate the set by using the set identities that we have above.

1.2.4.1 example

goal: prove that $\overline{A \cup (B \cap C)} = \left(\bar{C} \cup \bar{B}\right) \cap \bar{A}$

proof

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \bar{A} \cap \overline{(B \cap C)} && \text{demorgan's law} \\
&= \bar{A} \cap \left(\bar{B} \cup \bar{C}\right) && \text{demorgan again} \\
&= \left(\bar{B} \cup \bar{C}\right) \cap \bar{A} && \text{commutative law} \\
&= \left(\bar{C} \cup \bar{B}\right) \cap \bar{A} && \text{commutative law}
\end{aligned}$$