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1 sets

a **set** is an unordered collection of objects. each set can contain items of mixed types, and it can also contain other sets. the objects that make up the set are called **elements** of the set.

1.1 describing a set

there are three common ways to describe a set:

- explicit enumeration
  - $A = \{1, 2, 3, 4\}$
- ellipses if the pattern is trivial
  - $B = \{2, 4, 6, ..., 12\}$
- set builder notation
  - $C = \{y \mid y = 3k \text{ for some integer } k\}$
  - plain english: "this set `C` contains all elements y such that $y = 3k$ for some integer k."

1.2 notable sets

these are some common sets that are predefined and commonly used in mathematics:

$$
\begin{aligned}
&\mathbb{N} = \{0, 1, 2, 3, ...\} && \text{natural numbers} \\
&\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\} && \text{integers} \\
&\mathbb{Z}^+ = \{1, 2, 3, ...\} && \text{positive integers} \\
&\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\} && \text{rational numbers} \\
&\mathbb{R} && \text{real numbers} \\
&\emptyset = \{\} && \text{empty set}
\end{aligned}
$$

1.3 set equality

two sets are *equal* if and only if they contain exactly the same elements (excluding duplicated elements).

$$A = B \text{ if and only if } \forall x\,(x \in A \leftrightarrow x \in B)$$

1.4 subsets

some set A is a **subset** of another set `B` if and only if every element of A is an element in the set B. we write this as $A \subseteq B$, and call B a **superset** of A.

$$A \subseteq B \text{ iff } \forall x\,(x \in A \to x \in B)$$

1.4.1 proper subset

some A is a **proper subset** of B if and only if $A \subseteq B$ but $A \neq B$. we write this as $A \subset B$.

$$A \subset B \text{ iff } \forall x\,(x \in A \to x \in B) \land \exists y\,(y \in B \land y \notin A)$$

1.4.2 subset properties

- for all sets S, $\emptyset \subseteq S$.
- for any set S, $S \subseteq S$.
- if $S_1 = S_2$, then $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$.

1.5 set operations

we can perform operations between sets as well.

1.5.1 ∪ union

the **union** of two sets A and B contains every element that is either in A or in B. we write this as $A \cup B$.

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

we can also get the union of any number of set S by using a big $\cup$ operator, like this:

$$\bigcup_{i=1}^{n} S_i$$

1.5.2 ∩ intersection

the **intersection** of two sets A and B contains every element that is in A and also in B. we write this as $A \cap B$.

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

similar to union, we can use a big $\cap$ to get the intersection of any number of set S, like the following:

$$\bigcap_{i=1}^{n} S_i$$

we say that two sets are **disjoint** if $A \cap B = \emptyset$ (contains no overlapping elements).

1.5.3 – difference

the **difference** of two sets A and B contains every element that is in A but not in B. we write this as $A - B$.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

1.5.4 $\bar{A}$ complement

the complement of a set A, denoted by $\bar{A}$, contains every element that is in the universal set U, but not in A.

$$\bar{A} = \{x \mid x \in U \land x \notin A\}$$

1.5.5 $|S|$ cardinality

let S be a set. if there are exactly n elements in S, where n is a non-negative integer, then S is a finite set whose cardinality is n. the cardinality of S is denoted by $|S|$.

$$S = \{1, 2, 3, 4, 5\}$$
$$|S| = 5$$

1.6 power set

given a set S, its power set is the set containing all subsets of S. we denote the power set of S as $\mathcal{P}(S)$.

think of a set like a tree. we start with an empty set, and we start inserting items at the leaf node until it is the set we want. a power set of a set is like the every possible state that the tree could be in before it is what we have currently; see image.

example:

$$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$$
$$\mathcal{P}(\{1, 2, 3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

note:
- the set $\emptyset$ is in the power set of any set.
  - $\forall S \, (\emptyset \in \mathcal{P}(S))$
- the set S is in its own power set.
  - $\forall S(S \in \mathcal{P}(S))$
- the cardinality of the power set of any set S is 2 to the power of the cardinality of set S.
  - $|\mathcal{P}(S)| = 2^{|S|}$

1.6.1 power set of a nested set

for the power set of a nested set, you don't need to compute the power set of inner set; instead, treat it like a single item.

$$\mathcal{P}(\{1, 2, \{1, 2\}\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{\{1, 2\}\}, \{1, \{1, 2\}\}, \{2, \{1, 2\}\}, \{1, 2, \{1, 2\}\}\}$$

to check if you calculated the power set of a set correctly, you can check the cardinality; recall that the cardinality of the power set of any set S is $2^{|s|}$.

1.7 aside: ordered collections

the **ordered n-tuple** $(a_1, a_2, ..., a_n)$ is the *ordered* collection that has a_1 as its first element, a_2 as its second element, …, so on and so forth.

> **note**: $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., a_n)$ *if and only if* $a_i = b_i$ for $i = 1, ..., n$

1.8 cartesian product

if A and B are sets, the **cartesian product** of A and B (denoted as $A \times B$) is the set of all ordered pairs (a, b) such $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

the cartesian product of two sets are *not* commutative. that means, $(A \times B) \neq (B \times A)$.

1.8.1 cartesian product between multiple sets

example: let
- $S = \{x \mid x \text{ is enrolled in CS 441}\}$
- $G = \{x \mid x \in \mathbb{R} \land (0 \leq x \leq 100)\}$
- $Y = \{\text{freshman}, \text{sophomore}, \text{junior}, \text{senior}\}$

the set $S \times Y \times G$ consists of *all possible* $(\text{student}, \text{year}, \text{grade})$ combinations.

1.9 use with quantifiers

set notation allows us to make quantified statements more precise by explicitly stating the domain. for example, this statement:

$$\forall x \in \mathbb{R}\, (x^2 \geq 0)$$

states that the square of any real number is at least zero(0). let's try a more complex statement:

$$\forall n \in \mathbb{Z}\, \exists j, k \in \mathbb{Z}\, ((3n + 2 = 2j + 1) \rightarrow (n = 2k + 1))$$

states that if n is an integer and $3n + 2$ is odd, then n is odd.

1.9.1 truth sets

given a predicate $P(x)$ and its corresponding domain D, the **truth set** of $P(x)$ enumerates all elements in D that make the predicate P true. the truth set T_P (non canonical name) for any predicate $P(x)$ is denoted by:

$$T_P = \{x \in D \mid P(x)\}$$

note:
- $\forall x P(x)$ is true iff the truth set T_P is the *entire domain* D.
- $\exists x P(x)$ is true iff the truth set T_P is *non-empty*.

1.10 aside: bitmaps

we can represent sets as bitmaps, a common data structure used in computing. take the set $S = \{x \mid x \in \mathbb{N}, x < 10\}$; we can represent any subset of S just in $|S| = 10$ bits. before doing that however, we need to agree on an ordering. for ease of understanding, we use the natural order of numbers.

to represent any subset of S with a bitmap with natural ordering, we have 10 bits, with the n^{th} bit representing if the number n exists in the set. for example:

- $\{1, 3, 5, 7, 9\} = 0101\ 0101\ 01$
- $\{1, 1, 1, 4, 5\} = 0100\ 1100\ 00$ (we remove the duplicated elements in the set, so it is just $1, 4, 5$)