1 rules of inference

we use rules of inference to *formally construct valid arguments and proofs*. it is similar to **logical equivalence** where we rewrite and simplify single statements, we use **rules of inference** to combine multiple known truths (or **premises**) to deduct new knowledge.

1.1 terminology

- **conjecture**: a statement that is thought to be true.
- **proof**: a valid argument that establishes the truth of a statement (like a conjecture).
- **theorem**: a conjecture that is proven.
- **argument**: a sequence of statements (or **premises**) that ends with a conclusion.
- **rules of inference**: logically valid ways to draw new conclusions by combining known truths from a **tautology**.

a **formal proof** consists of a sequence of statements, starting with the premises (axioms), and ending with a conclusion; where each step logically follows from the preceding ones.

1.2 key rules of inference

these are the most often used rules of inference. each one consists of a tautology, logical form, and a given example in plain words.

1.2.1 modus ponens (law of detatchment)

if $p \to q$ is true and p is also true, then q must be true.

$$\text{rule: } p \to q, p \vdash q \qquad \text{tautology: } (p \land (p \to q)) \to q$$

- **premise 1**: if it is raining (p), then the grass will be wet (q).
- **premise 2**: it is raining p.
- **conclusion**: therefore, the grass is wet ($\therefore q$).

1.2.2 modus tollens (the law of contrapositive)

it is like the reverse of **modus ponens**, if $p \to q$ is true, but q is false, then p must be false.

$$\text{rule: } p \to q, \neg q \vdash \neg p \qquad \text{tautology: } (\neg q \land (p \to q)) \to \neg p$$

- **premise 1**: if i have a cat (p), then i buy cat food (q).
- **premise 2**: i did not buy cat food ($\neg q$).

- **conclusion**: therefore, i do not have a cat ($\therefore \neg p$).

1.2.3 hypothetical syllogism (chain rule)

we can combine two *if … then …* chains together.

$$\text{rule: } p \to q, q \to r \vdash p \to r \qquad \text{tautology: } ((p \to q) \land (q \to r)) \to (p \to r)$$

- **premise 1**: if i study hard (p), then i will get a good grade (q).
- **premise 2**: if i get a good grade (q), then my parents will be happy (r).
- **conclusion**: therefore, if i study hard (p), then my parents will be happy (r). ($\therefore p \to r$)

1.2.4 disjunctive syllogism (elimination)

in an *or* statement, for the whole statement to be true, if one choice is false, then the other must be true.

$$\text{rule: } p \lor q, \neg p \vdash q \qquad \text{tautology: } ((p \lor q) \land \neg p) \to q$$

- **premise 1**: i will eat cake (p) or i will eat ice cream (q).
- **premise 2**: i will not eat cake ($\neg p$).
- **conclusion**: therefore, i will eat ice cream ($\therefore q$).

1.2.5 addition

if something is true, you can make a bigger *or* statement with it, and it will still be true.

$$\text{rule: } p \vdash p \lor q \qquad \text{tautology: } p \to (p \lor q)$$

- **premise 1**: the sky is blue (p).
- **conclusion**: therefore, the sky is blue (p) or unicorns are real (q). ($\therefore p \lor q$)

1.2.6 simplification

if two things are true together, then each one must be true by itself.

$$\text{rule: } p \land q \vdash p \qquad \text{tautology: } (p \land q) \to p$$

- **premise 1**: it is sunny (p) and it is warm (q).
- **conclusion**: it is sunny ($\therefore p$).

1.2.7 conjunction

if two separate things are true, we can combine them with *and*. it is the opposite of **simplification**.

$$\text{rule: } p, q \vdash p \wedge q \qquad \text{tautology: } ((p) \wedge (q)) \to (p \wedge q)$$

- **premise 1**: i have a pencil (p).
- **premise 2**: i have paper (q).
- **conclusion**: therefore, i have a pencil (p) and paper (p). ($\therefore p \wedge q$)

1.2.8 resolution

we use resolution to combine two *or* statements and cancel out the conflicting part.

$$\text{rule: } p \vee q, \neg p \vee r \vdash q \vee r \qquad \text{tautology: } ((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$$

- **premise 1**: we will go to the park (p) or we will go to the cinema (q).
- **premise 2**: we will not go to the park ($\neg p$) or we will buy popcorn (r).
- **conclusion**: ignore the conflicting part, we will go to the cinema (q) or we will buy popcorn (r).

1.3 rules of inference for quantified statements

we also have rules of inferences for statements with $\forall$ and $\exists$. for each quantifier, we have a **instantiation** rule and **generalization** rule. in simple words, **instantiation** is going from a general idea to a specific example; and **generalization** is going from specific examples to a general idea.

1.3.1 universal instantiation (ui)

if something is true for *everything* in a domain, then it must be true for *any specific one* you pick from it.

$$\exists x P(x) \vdash P(c)$$

- **premise**: all cats ($\forall x$) are cute ($P(x)$).
- **conclusion**: therefore, my cat whiskers (c) is cute ($P(c)$).

1.3.2 universal generalization (ug)

if something is true for a totally *arbitrary and generic* individual, then it is true for *everything*. the individual can't have any special properties.

$$P(c) \text{ for an arbitrary } c \vdash \forall x P(x)$$

- **premise**: let n be any integer whatsoever. n must be even or odd.
- **conclusion**: since n does not have any special property, all integers $\forall x \in \mathbb{Q}$ are either even or odd.

1.3.3 existential instantiation (ei)

if there exists at least one thing with a special property, we can give it a temporary (new) name so we can reason with it.

$$\exists x P(x) \vdash P(c) \text{ for some new element } c$$

- **premise**: someone $\exists x$ stole the cookie ($P(x)$).
- **conclusion**: we can call this person "cookie snatcher" (c), and cookie snatcher stole the cookie ($P(c)$).

1.3.4 existential generalization (eg)

if you can find just one specific example of something, you can say that there exists at least one of them.

$$P(c) \vdash \exists x P(x)$$

- **premise**: my cat whiskers (c) have black fur ($P(c)$).
- **conclusion**: therefore, there exists at least one cat ($\exists x$) with black fur ($P(x)$).