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1 predicates and quantifiers

propositional logic is simple, however limited. we can't use propositional logic to represent some classes of natural language statements such as this:

given that:
- all of my dogs like peanut butter, and
- kody is one of my dogs

can we draw the conclusion that kody likes peanut butter with propositional logic?

another example, if we want to reason with prime numbers:
- some natural numbers are prime numbers
- a prime number has no divisors other than 1 and itself

in order to draw the conclusion with what we have learned with propositional logic so far, we will have to enumerate all the prime numbers:
- $p_1 \equiv 2$ has no divisors other than 1 and itself
- $p_2 \equiv 3$ has no divisors other than 1 and itself
- $p_3 \equiv 5$ has no divisors other than 1 and itself
- $p_4 \equiv 7$ has no divisors other than 1 and itself
- $p_5 \equiv 11$ has no divisors other than 1 and itself
- ...

the previous examples are called *syllogisms* – where a conclusion can be drawn from two given or assumed propositions. for example:

$$\begin{array}{ll} \text{All men are mortal} & \text{(major premise)} \\ \text{Socrates is a man} & \text{(minor premise)} \\ \hline \therefore \text{Socrates is mortal} & \text{(conclusion)} \end{array}$$

we can use predicates and quantifiers to establish syllogisms, similar to the one shown above.

1.1 predicates

predicate logic allows us to use *propositional functions* during our logical reasoning. for example:

$$P(x) \equiv x^3 > 0$$

in this propositional function, x is the variable, and $x^3 > 0$ is the predicate. note that propositional function $P(x)$ *has no truth value* unless it is evaluated for a given x or a set of xs. for example:

$$P(0) \equiv \bot$$
$$P(23) \equiv \top$$
$$P(-42) \equiv \bot$$

The predicate doesn't have to be just one variable. These are also valid predicates:

$$P(x, y) \equiv x + y = 42$$
$$S(x, y, z) \equiv x + y = z$$

1.2 quantifiers

with *quantifiers*, we can make general statements that turn propositional functions into propositions. in english, we use quantifiers regularly:
- **all** students can ride the bus for free
- **many** people like chocolate
- i enjoy **some** types of tea
- **at least one** person will sleep through their final exam

quantifiers require us to define a *universe of discourse* (or a *domain*) in order for the quantification to make sense.

1.2.1 universal quantifier

a **universal quantifier** allows us to make statements about the *entire domain*. for example:
- **all** of my dogs like peanut butter
- **every** even integer is a multiple of two
- **for each** positive integer x, $2x > x$

in mathematical notation, we can express the universal quantification of $P(x)$ as $\forall x\, P(x)$.

remember the second example in section 1? take the proposition: "if a natural number is prime, it has no divisors other than 1 and itself." we can rewrite it using predicates and quantifiers:

$$P(x) \equiv x \text{ is prime}$$
$$D(x) \equiv x \text{ has no divisors other than 1 and itself}$$
$$\boxed{\forall x \in \mathbb{N}\ [P(x) \to D(x)]}$$

1.2.2 existential quantifier

an **existential quantifier** allow us to make statements about *some objects* in a domain. for example:
- **some** elephants are scared of mice
- **there exist** integers a, b, and c such that the equality $a^2 + b^2 = c^2$ is true
- **there is at least one** person who did better than john on the midterm

given a propositional function $P(x)$, we express the existential quantification of $p(x)$ as $\exists x\, p(x)$.

for example: "the inequality x + 1 < x holds for at least one integer" can be written in the following:

$$P(x) \equiv x + 1 < x$$
$$\exists x \in \mathbb{Q}\ [P(x)]$$

1.2.3 domain restriction

we can also restrict the domain of quantification. for example, take this statement: "the square of every natural number less than 4 is no more than 9"
- **domain**: natural numbers ($\mathbb{N}$)
- **statement**: $\forall x < 4\ (x^2 \leq 9)$
  - this is equivalent to $\forall x\ [(x < 4) \to (x^2 \leq 9)]$
- **truth value**: true

this also works for the existential quantifier; take this statement: "some integers between 0 and 6 are prime"
- **domain**: integers ($\mathbb{Q}$)
- **statement**: $\exists 0 \leq x \leq 6\ P(x)$
  - equivalent to $\exists x\ [(0 \leq x \leq 6) \land P(x)]$
- **truth value**: true

1.3 quantifiers precedence

the universal and existential quantifiers have the *highest precedence* of all logical operators. when needed, use parentheses to disambiguate a quantifier's scope.
- $\forall x\, P(x) \to Q(x)$ actually means $(\forall x\, P(x)) \to Q(x)$

- $\exists x\, P(x) \wedge Q(x)$ actually means $(\exists x\, P(x)) \wedge Q(x)$

1.4 note on logical equivalence

two statements involving predicates and quantifiers are **logically equivalent** *if and only if* they take on the same truth value regardless of which predicates are substituted into these statements and which domains of discourse are used.

1.5 negating quantifiers

1.5.1 universal quantifier

negating a universal quantifier will turn it into a existential quantifier and negate the propositions inside it.

$$\neg(\forall x\, P(x)) \equiv \exists x\, (\neg P(x))$$

1.5.2 existential quantifier

negating is similar for existential quantifier. $\exists$ turn into $\forall$ and negate the propositions inside it.

$$\neg(\exists x\, P(x)) \equiv \forall x\, (\neg P(x))$$