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1 logical puzzles

1.1 example 1

alice and bob are technical support agents. if an agent is having a bad day, they will always lie to you. if an agent is having a good day, they will always tell you the truth. alice tells you that bob is having a bad day. bob tells you that he and alice are both having the same type of day. can you trust the advice you receive from alice during your call?

1.1.1 rules

- if an agent is having a bad day, they will always lie
- if an agent is having a good day, they will always tell the truth
- they can't be both lying or telling the truth at the same time

1.1.2 propositions

- $A \equiv \text{alice is having a bad day}$
- $B \equiv \text{bob is having a bad day}$
- $\neg A \to B$
  - if alice isn't lying, then bob is having a bad day
- $\neg B \to (A \leftrightarrow B)$
  - if bob isn't lying, then alice and bob are both having the same type of day

1.1.3 case 1

we assume A is true, where alice is having a bad day. that means alice lies about bob having a bad day; therefore bob is having a good day.

based on alice's statement, we can deduct that $A \equiv \top$ and $B \equiv \bot$. if we substitute in the truth values in the propositions above, we can get that:

$$\neg\top \to \bot = \top \quad \checkmark$$
$$\neg\bot \to (\top \leftrightarrow \bot) = \top \to \bot = \bot \quad \times \text{ (contradiction)}$$

1.1.4 case 2

we assume B is true, where bob is having a bad day, therefore, according to the rules, alice must be having a good day, where $A \equiv \bot$ and $B \equiv \top$. we can then substitute in the values:

$$\neg\top \to (\bot \leftrightarrow \top) = \bot \to \bot = \top \quad \checkmark$$
$$\neg\bot \to \top = \top \to \top = \top \quad \checkmark$$

1.1.5 conclusion

since case 2 is the only case that satisfies both propositions, we can conclude that bob is having a bad day, and alice is having a good day; therefore, we can trust alice's advice.

1.2 example 2

consider a group of friends: fredrik, anuradha, and cai. if fredrik is not the oldest, then anuradha is. if anuradha is not the youngest, then cai is the oldest. determine the relative ages of fredrik, anuradha, and cai.

1.2.1 rules

- fredrik, anuradha and cai can't be the the oldest or the youngest at the same time

1.2.2 propositions

- $F \equiv \text{fredrik is the oldest}$
- $A \equiv \text{anuradha is the oldest}$
- $A' \equiv \text{anuradha is the youngest}$
- $C \equiv \text{cai is the oldest}$
- $\neg F \to A$
- $\neg A' \to C$
- $\neg(A \land F)$
- $\neg(A \land C)$
- $\neg(C \land F)$

1.2.3 case 1

1.2.4 case 2

2 propositional equivalence

2.1 definitions

tautology: a compound proposition that is **always true**, regardless of the truth values of the propositions occurring within it. ($p \lor \neg p$)

contradition: a compound proposition that is **always false**, regardless of the truth values of the propositions occurring within it. ($\neg p \land p$)

contingency: a compound proposition whose truth value is **dependent** on the propositions occurring within it. ($p \lor q$)

logically equivalent: two propositions p and q are logically equivalent exactly when $p \leftrightarrow q$ is a *tautology*. we use the notation $p \equiv q$ to express that p and q are logically equivalent. ($p \to q \equiv \neg p \lor q$)

2.2 proving simple logical equivalences

for simple propositions, we can just create a truth table for both sides. for example, take this equivalence: $p \to q \equiv \neg p \lor q$. we only need to create 4 rows, since there is only p and q.

p, q	$\neg p$	$\neg p \lor q$	$p \to q$
$\bot\bot$	$\top$	$\top$	$\top$
$\bot\top$	$\top$	$\top$	$\top$
$\top\bot$	$\bot$	$\bot$	$\bot$
$\top\top$	$\bot$	$\top$	$\top$

however, since the number of rows grow exponentially(2^n), this is not usually the best way to prove an equivalence. in the next section, we introduce some common yet important logic equivalences to help you construct new logical equivalences.

2.3 useful logic equivalences

we use T to denote a compound proposition that is a **tautology** and F for a compound proposition that is a **contradiction**.

name	equivalence
identity laws	$p \land T \equiv p$ $p \lor F \equiv p$

name	equivalence
domination laws	$p \land F \equiv F$ $p \lor T \equiv T$
idempotent laws	$p \land p \equiv p$ $p \lor p \equiv p$
double negation law	$\neg(\neg p) \equiv p$
commutative laws	$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$
associative laws	$(p \land q) \land r \equiv p \land (q \land r)$ $(p \lor q) \lor r \equiv p \lor (q \lor r)$
distributive laws	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$ $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
demorgan's laws	$\neg(p \lor q) \equiv \neg p \land \neg q$ $\neg(p \land q) \equiv \neg p \lor \neg q$
absorption laws	$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$
negation laws	$p \lor \neg p \equiv T$ $p \land \neg p \equiv T$

2.4 other logical equivalences

conditional logic equivalences
$p \to q \equiv \neg p \lor q$ $p \to q \equiv \neg q \to \neg p$ $p \lor q \equiv \neg p \to q$ $p \land q \equiv \neg(p \to \neg q)$ $\neg(p \to q) \equiv p \land \neg q$ $(p \to q) \land (p \to r) \equiv p \to (q \land r)$ $(p \to r) \land (q \to r) \equiv (p \lor q) \to r$ $(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$ $(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

biconditional logic equivalences
$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

3 activities

3.1 question 1: prove that $(p \land q) \to (p \lor q)$ is a tautology.

$$
\begin{aligned}
&(p \land q) \to (p \lor q) \\
&\equiv \neg(p \land q) \lor (p \lor q) \\
&\equiv \neg p \lor \neg q \lor p \lor q \\
&\equiv (\neg p \lor p) \lor (\neg q \lor q) \\
&\equiv \top \lor \top \\
&\equiv \top
\end{aligned}
$$

3.2 question 2: prove that $(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$$
\begin{aligned}
&(p \to q) \lor (p \to r) \\
&\equiv \neg p \lor q \lor \neg p \lor r \\
&\equiv \neg p \lor q \lor r \\
&\equiv \neg p \lor (q \lor r) \\
&\equiv p \to (q \lor r)
\end{aligned}
$$