1 infinite cardinality

a finite set, or a set that has the same cardinality as $\mathbb{N}$ (or $\mathbb{Z}^+$) is called **countable**. otherwise, it is called **uncountable**.

1.1 countables

natural numbers have the same cardinality as positive integers. take this function: $f : \mathbb{N} \to \mathbb{Z}^+, f(x) = x + 1$; where it maps natural numbers to positive integers. first, it is *surjective*: every positive integer k in the codomain is mapped by a natural number $k - 1$. second, no two natural numbers have the same mapping, therefore it is *injective* (one-to-one). therefore, it is a *bijection*, implying that the two cardinalities are the same: $|\mathbb{N}| = |\mathbb{Z}^+|$, and both sets are countably infinite.

what about for $\mathbb{Z}$? since it contains negative integers, it should be twice as large, right? let's look at the proof:

$$f : \mathbb{Z} \to \mathbb{Z}^+, f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$$

this function maps all integers to positive integers. we can see that all elements in the codomain are mapped mapped by interleaved positive/negative integers, therefore a *surjection*; we can also see that no two integer numbers have the same mapping (one-to-one, *injection*). since it is both surjective and injective, it is bijective; implying that the cardinalities are the same ($|\mathbb{Z}| = |\mathbb{Z}^+|$), and both sets are countably infinite.

1.2 uncountables

the set of real numbers $\mathbb{R}$ is uncountable.