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1 summations

we can use summation to find the sum of a sequence of numbers. a **sequence** is a function from a subset of of the set of integers to a set S. we use the notation a_n to denote the image of the integer n. a_n is called the **term** of a sequence.

1.1 special sequences

geometric progressions are sequences of the form $\{ar^n\}$ where $a, r \in \mathbb{R}$.
- $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ ($a = 1, r = \frac{1}{2}$)
- $1, -1, 1, -1, 1, ...$ ($a = 1, r = -1$)

arithmetic progressions are sequences of the form $\{a + nd\}$ where $a, d \in \mathbb{R}$.
- $2, 4, 6, 8, 10, ...$ ($a = 2, d = 2$)
- $-10, -15, -20, -25$ ($a = -10, d = -5$)

1.2 recurrence

sequences are often specified using recurrence relations (recursive approach); where later terms are specified from earlier terms.

for example, the fibonacci sequence:
- $f_0 = 0$
- $f_1 = 1$
- for any $n > 1$, $f_n = f_{n-1} + f_{n-2}$

to write a recursive sequence, we need at least one **initial condition** (or base case).

1.3 summation notation

the summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + ... + a_n$.

$$\sum_{j=m}^{n} a_j = \sum_{m \leq j \leq n} a_j$$

j is the index of summation, while m and n are the lower and upper limit, respectively.

1.3.1 summation operations

the summation operation is linear; the usual laws of algebra apply. constant factors can be pulled out of the summation, and a summation over a sum (or difference) can be split into a sum (or difference) of smaller summations. example:

$$\sum_{j=1}^{n}(dx_j + by_j - cz_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j - c\sum_{j=1}^{n} z_j$$

1.3.2 summation over sets

summations can also be used to find the sum (with some operation) over a set, it doesn't have to be just a sequence. for example:

$$\begin{aligned} &\sum_{s\in\{0,2,4,6\}}(s+2) \\ &= (0+2)+(2+2)+(4+2)+(6+2) \\ &= 20 \end{aligned}$$

1.3.3 nested summations

summations can also be nested. often, we will see this when analyzing nested loops within a program. example:

$$\begin{aligned} &\sum_{j=1}^{4}\sum_{k=1}^{3}(jk) \\ &= \sum_{j=1}^{4}(j+2j+3j) && \text{expand inner sum} \\ &= \sum_{j=1}^{4}6j && \text{simplify} \\ &= 6+12+18+24 && \text{expand outer sum} \\ &= 60 \end{aligned}$$

1.4 reorganizing summations

sometimes it is helpful to shift the index of the summation, so we can combine two or more similar summations. see example:

$$\begin{aligned}
s &= \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k-1) \\
&= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j+1)-1) \\
&= \sum_{j=1}^{10} (j^2 + 2(j+1) - 1) \\
&= \sum_{j=1}^{10} (j^2 + 2j + 1) \\
&= \sum_{j=1}^{10} (j+1)^2
\end{aligned}$$

in the second step, we changed the index of the summation; we also changed the lower and upper limit. since we moved the summation's range by 1, we need to add 1 to j.

1.5 closed forms

calculating the sum of series can be time consuming. thankfully, there are closed forms of certain series that can help us calculate the sum of them rather trivially.

this is the closed form for a geometric series:

$$\sum_{j=0}^{n} = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

here are some other sums that have its corresponding closed form:

sum	closed form
$\sum_{j=1}^{n} j$	$\frac{n(n+1)}{2}$
$\sum_{j=1}^{n} j^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{j=1}^{n} j^3$	$\frac{n^2(n+1)^2}{4}$

1.5.1 aside: proof of geometric series' closed form

$$
\begin{aligned}
S_n &= \sum_{j=0}^{n} ar^j \\
r \cdot S_n &= r \cdot \sum_{j=0}^{n} ar^j && \text{multiply by r on both sides} \\
r \cdot S_n &= \sum_{j=0}^{n} ar^{j+1} \\
r \cdot S_n &= \sum_{k=1}^{n+1} ar^k && \text{substitute k = j+1} \\
r \cdot S_n &= \sum_{k=1}^{n} ar^k + ar^{n+1} && \text{take out last element} \\
r \cdot S_n &= \sum_{k=1}^{n} ar^k + ar^{n+1} && \text{take out last element} \\
r \cdot S_n &= \sum_{k=0}^{n} ar^k - ar^0 + ar^{n+1} && \text{move lower and subtract extra} \\
r \cdot S_n &= S_n - a + ar^{n+1} && \text{substitute sum with } S_n \\
r \cdot S_n - S_n &= ar^{n+1} - a \\
S_n(r-1) &= ar^{n+1} - a \\
S_n &= \frac{ar^{n+1} - a}{r-1} && \blacksquare
\end{aligned}
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