1 functions

sets give us a way to formalize the concept of a function. take two non-empty sets for example, A and B. a **function** f is an assignment of *exactly one* element of set B to each element of set A.

we write $f : A \to B$ to denote that f is a function from set A to set B; and we say that $f(a) = b$ if the element $a \in A$ is mapped to the unique element $b \in B$ by the function f.

1.1 defining a function

a function can be defined in a number of ways:

1. explicitly
   - $f : \mathbb{Z} \to \mathbb{Z}$
   - $f(x) = x^2 + 2x + 1$
2. using a programming language
   - `const min: number = (x: number, y: number) => x < y ? x : y`
   - `int max(int x, int y) = {x > y ? return x : return y}`
3. using a relation
   - let $S = \{\text{Anna}, \text{Brian}, \text{Christine}\}$
   - let $G = \{A, B, C, D, F\}$
   - let $f : S \to G$.
     $f(\text{Anna}) = C$
     $f(\text{Brian}) = A$
     $f(\text{Christine}) = A$

1.2 terminology

- the **domain** of a function is the set that a function maps *from* ($f : \boldsymbol{A} \to B$).
- the **codomain** of a function is the set that a function maps *to* ($f : A \to \boldsymbol{B}$).
- take the function $f(a) = b$. b is the **image** (output) of a, and a is the **preimage** (input) of b.
- the **range** of a function $f : A \to B$ is the set of *all images* of elements of a (or all possible outputs).

1.3 types of functions

1.3.1 injective

a **one to one**, or **injective** function is a function that never assigns the same image to two different elements. a function is **injective** if and only if:

$$\forall x, y \in A\ [(f(x) = f(y)) \to (x = y)]$$

1.3.2 surjective

we call a function $f : A \to B$ **onto**, or **surjective**, if and only if for every element $b \in B$, there is some element $a \in A$ such that $f(a) = b$. we can write this formally:

$$\forall y \in B, \exists x \in A\ [f(a) = b]$$

1.3.3 bijective

functions that are both *injective* and *surjective* are called **bijections** (or **bijective**). you can formulate the formal definition yourself; it is just the two definitions above conjuncted.

bijections are the only functions that can have an **inverse**. the *injective-ness* make sure that for each preimage of the inverse function, there is **only one** image; the *surjective-ness* makes sure that there is at least one image for preimage, so the entire domain must have a corresponding output.

in short: if $f : A \to B$ is a bijection, then the inverse of f is the function $f^{-1} : B \to A$ that assigns to each $b \in B$ the unique value $a \in A$ such that $f(a) = b$.

1.4 function composition

functions can be composed with one another. given a function $f : A \to B$ and $g : B \to C$, the **composition** of f and g, is denoted as $g \circ f$, and $(g \circ f)(x) = g(f(x))$.

for two functions to compose, the codomain of the inner function ($f(x)$) must be a subset of the domain of the outer function ($g(x)$).

1.5 important functions

1.5.1 floor

the **floor** function maps a real number $x \in \mathbb{R}$ to the *largest* integer $y \in \mathbb{Z}$ so that y is *not greater* than x. The **floor** of x is denoted as $\lfloor x \rfloor$.

1.5.2 ceiling

The ceiling function maps a real number $x \in \mathbb{R}$ to the *smallest* integer $y \in \mathbb{Z}$ so that y is *not less* than x. The ceiling of x is denoted $\lceil x \rceil$.